ACME

Advisory Committee on Mathematics Education

Problem solving in mathematics: realising the vision through better assessment

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1 Introduction

Problem solving is an important component of mathematics across all phases of education.

In the modern world, young people need to be able to engage with and interpret data and information. They need to become flexible thinkers capable of dealing with novel problems and situations and analysing their own and others' solutions to these. ACME has long recognised the centrality of problem solving in the effective teaching, learning and assessment of mathematics and this has been reflected in its policy work, as set out in Box 1.

The current education system prioritises assessment and accountability. This means that assessment will impact on how mathematics is taught in the classroom. Therefore, high-quality assessment of problem solving in public tests and assessments[1] is essential in order to ensure the effective learning and teaching of problem solving throughout primary and secondary education. Although the focus here is on the assessment of problem solving in mathematics, many of the ideas will be directly transferable to the assessment of problem solving in other subjects, e.g. in the sciences and beyond.[2]

Problem solving is a broad encompassing term and providing a single prescriptive definition is of limited merit.[3] For the purposes of this report it is more useful to consider the desirable characteristics of questions used to assess problem solving. In this report ACME:

- considers the assessment of problem solving in public tests and examinations across all key stages of mathematical development in the light of recent reforms (section 2);
- describes the desirable characteristics of questions used to assess problem solving (section 3);
- sets out actions for policymakers, awarding organisations and the mathematics community to ensure that improvements in the quantity and quality of problem solving in mathematics tests and assessments are realised over time (section 4).

Box 1: Summary of ACME's recent work on problem solving

Since 2010 ACME's work on problem solving has included:

- a call for exemplars of effective assessment of problem solving in mathematics in spring 2015;
- a series of conference workshops and discussion sessions on problem solving across all phases from 2013 – 2015;
- policy advice on recent reforms across all key stages;
- sections in ACME reports, for example *Mathematical Needs: Mathematics in the workplace and in Higher Education* and *Mathematical Needs of learners*.

Further information can be found at **www.acme-uk.org/policy-advice/qualifications-and-assessment**.

1 The focus in this report is on summative assessment, that is assessment that sums up what has been achieved at the end of a period of time, relative to the learning aims and national standards.

2 Many new A level subjects also have specific mathematical requirements. See ACME and the Royal Statistical Society (RSS) 2015 *Embedding Statistics at A level: a report on statistical requirements and assessment across A level courses in Biology, Business, Chemistry, Geography, Psychology and Sociology* (see http://www.acme-uk.org/media/32719/embedding-statistics-at-a-level.pdf accessed 25 April 2016).

3 Schoenfeld, A. H. 1992, 'Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics' in D. Grouws (Ed.) *Handbook for Research on Mathematics Teaching and Learning* New York: MacMillan (pp 334 – 370).

2 Problem solving in mathematics assessment

Mathematics assessment should reflect what the mathematics community, employers and universities value about the subject.

There is agreement that young people need to leave education able to use and apply mathematics at a level appropriate for their chosen careers.[4] For this to happen, assessment across all phases of education must provide learners with substantial questions, rich in challenge and problem-solving opportunity. Problem solving is something that all learners should experience. In this context it is important to realise that a challenging problem solving question at a particular phase of learning does not necessarily require difficult mathematics.

In recent years there has been a great deal of reform to mathematics curricula, from early years through to A level, with problem solving forming an important part of the aims of all public tests and examinations. (Table 1 gives a summary of recent reforms across phases.)

Recent history demonstrates that while problem solving has been part of expectations in official mathematics curricula and qualification documents, this has not generally translated into meaningful assessment of problem solving.[5] The emphasis on problem solving in recent reforms to mathematics curricula is a positive development. However, the importance attached to problem solving must also extend to assessment in order to drive improvements in the teaching and learning of problem solving.

4 A level Content Advisory Board (ALCAB) 2014 *Report of the ALCAB Panel on Mathematics and Further Mathematics* (see https://alevelcontent.files.wordpress.com/2014/07/alcab-report-on-mathematics-and-further-mathematics-july-2014.pdf accessed 25 April 2016). CBI 2010 *Making it all add up: Business priorities for numeracy and maths* (see http://www.cbi.org.uk/media/935352/2010.08-making-it-all-add-up.pdf, accessed 25 April 2016). Higher Education Academy (HEA) 2014 *Mathematical transitions: a report on the mathematical and statistical needs of students undertaking undergraduate studies in various disciplines* (see https://www.heacademy.ac.uk/sites/default/files/resources/hea_mathematical-transitions_webv2.pdf, accessed 25 April 2016).

5 The quality and extent of problem solving in public examinations has been substantially diminished over the last 50 years. This reduction began in the 1960s at A level. It continued through several adjustments to qualifications, which included the introduction of GCSE in 1988 and subsequent revisions to pre- and post-16 qualifications.

Table 1: An overview of some recent reforms		
Key stage	**First teaching of new qualifications**	**Problem solving in official documents***
Key Stages 1 – 3 Mathematics	New National Curriculum, first teaching September 2014[6]	One of the three aims is that all learners can 'solve problems by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication, including breaking down problems into a series of simpler steps and persevering in seeking solutions'.
GCSE Mathematics	New GCSE Mathematics qualification, first teaching September 2015[7]	It is set out that 'GCSE Mathematics must enable students to acquire, select and apply mathematical techniques to solve problems'.
AS/A level Mathematics	New AS/A level Mathematics, first teaching September 2017[8]	It is set out that qualifications must encourage learners to 'use their mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly'.
Core Maths	New suite of Core Maths qualifications, first teaching September 2015[9]	In the Core Maths technical guidance it is stated that the qualifications should 'consolidate and build on students' mathematical understanding and develop further mathematical understanding and skills in the application of maths to authentic problems'.

* This column sets out extracts from Department for Education programmes of study and subject specification documents, in particular from aims and objectives sections. Some of the documents have more detailed statements on problem solving than others.

ACME welcomes the research undertaken by Ofqual to ensure that assessment encourages problem solving in the new GCSE Mathematics qualification.[10] The assessment of problem solving was also prioritised during the reform of A level Mathematics and Further Mathematics through the work of the A level Content Advisory Board (ALCAB)[11] and the Ofqual A level Mathematics Working Group.[12] However, there has been a notable lack of prioritisation of problem solving in recent Key Stage 1 and 2 assessment materials.[13] This will have a detrimental effect on learners' development during primary schooling, and will inevitably impact on their preparedness for Key Stage 3 and GCSE Mathematics. Greater continuity in qualifications and curricula across all phases of mathematical education is required if sustained progress is to be realised (see section 4 for more detail).

6 Department for Education 2013 *Mathematics programmes of study* (see https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study, accessed 25 April 2016).

7 Department for Education 2014 *Mathematics GCSE subject content and assessment objectives* (see https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/254441/GCSE_mathematics_subject_content_and_assessment_objectives.pdf, accessed 25 April 2016).

8 Department for Education 2016 *Mathematics AS and A level content* (see https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/516949/GCE_AS_and_A_level_subject_content_for_mathematics_with_appendices.pdf, accessed 25 April 2016).

9 Department for Education 2015 *Core maths qualifications: technical guidance* (see https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/450294/Core-Maths-Technical-Guidance.pdf, accessed 25 April 2016).

10 Ofqual 2015 GCSE maths: *Final research report and regulatory summary* (see https://www.gov.uk/government/publications/gcse-maths-final-research-report-and-regulatory-summary, accessed 25 April 2016).

11 See https://alcab.org.uk/.

12 Ofqual 2015 *A Level Mathematics Working Group Report on Mathematical Problem Solving, Modelling and the Use of Large Data Sets in Statistics in AS/A Level Mathematics and Further Mathematics* (see https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/481857/a-level-mathematics-working-group-report.pdf, accessed 25 April 2016).

13 Standards and Testing Agency 2015 *National curriculum assessments: 2016 sample materials* (see https://www.gov.uk/government/collections/national-curriculum-assessments-2016-sample-materials, accessed 25 April 2016).

3 The assessment of problem solving

At their best, summative assessments in mathematics should provide learners with rich and diverse opportunities for problem solving.

The nature of the problem solving will vary considerably depending on the phase of mathematical development and the mathematical content being assessed. However, irrespective of these, the effective assessment of problem solving depends critically on the actual questions used to assess problem solving.[14]

3.1 Problem solving assessment design

In Table 2 ACME sets out a list of desirable characteristics of questions used to assess problem solving.[15] The list does not set out to be exhaustive, but aims to capture key recurring themes that arise repeatedly in discussions on the effective assessment of problem solving. Questions designed to assess problem solving within tests and examinations should be expected to contain some, but not necessarily all, of these characteristics.

Table 2: Characteristics of questions to assess problem solving

Varying the presentation	• Questions set within unfamiliar contexts or formats • Questions based on authentic scenarios [16] • Questions which are open-ended • Questions requiring the translation of text into mathematical forms
Making choices	• Questions in which a method of approach is not immediately obvious • Questions which can be solved by a range of methods • Questions requiring the selection of relevant information • Questions with multiple steps but little or no scaffolding [17]
Thinking mathematically	• Questions requiring abstract thinking • Questions requiring assumptions to be made • Questions requiring movement between mathematical representations e.g. numerical, graphical, diagrammatic, algebraic • Questions requiring the synthesis of mathematical ideas or approaches
Obtaining results	• Questions leading to a range of different possible solutions [18] • Questions involving the interpretation of solutions • Questions requiring the communication of solutions
Making modifications	• Questions enabling the critical analysis of solutions • Questions requiring the evaluation of solutions • Questions in which information can be revised • Questions in which approaches can be refined

14 In this context a question may contain one or more constituent parts.

15 To consider how best to assess problem solving at A level Ofqual's A Level Mathematics Working Group set out some possible attributes of assessment problem solving tasks. See reference 12.

16 Authentic tasks are 'based on situations which, while sometimes fictional, represent the kinds of problem encountered in real life', OECD 2001 *Knowledge and skills for life. First results from the OECD Programme for International Student Assessment (PISA) 2000* Paris: OECD.

17 In mathematics a question with scaffolding is one which contains intermediate prompts that provide guidance on the steps to be taken. Sometimes this is achieved by dividing a question into multiple parts which provide direction on the steps to be followed.

18 In this report the word 'solution' is used to encompass the wide range of possible outcomes from a problem solving question e.g. mathematical results or findings.

3.2 What do the characteristics of questions mean in terms of assessment and test design?

Avoiding predictability: At all stages of mathematics education the avoidance of predictability is key to the design of questions to assess problem solving. Therefore, the way in which the problem solving question is presented in assessment is important. The value in terms of problem solving will be diminished if, for example: (1) the task within the question is very familiar to the student; (2) the mathematical methods are identified explicitly in the question; (3) the question is highly scaffolded.

Building in choices: Choosing appropriate methods of approach is an important part of the problem solving process. Many problem solving questions require the translation of information given as text into a mathematical form so that mathematical techniques can be applied. In some cases there may be alternative mathematical forms and the choices made can impact substantially on the eventual method of solution.

Drawing conclusions: Mathematical solutions may have to be interpreted and communicated to provide answers to specific questions. The application of mathematical methods within a problem solving question will usually lead to some solution (or solutions) which may take different forms e.g. a numerical value or a mathematical result. Such solutions may need to be interpreted and communicated back within the original context in order to answer the problem solving question.[19] This stage may involve critically evaluating the mathematical solution(s) and/or the rejection of one solution in preference to another.

Probing mathematical thinking: Problem solving questions can sometimes test mathematical thinking by probing closely-related variants of the original question. Making variations to the information in a problem solving question, or changing the assumptions made in order to solve it, can lead to different solutions. Sometimes these revised solutions can be predicted from the solution to the original problem solving question using mathematical reasoning, without having to repeat the full problem solving process.

3.3 What do the characteristics of questions mean in terms of mark schemes?

Including more questions with the characteristics given in Table 2 will have implications for the design of mathematics examinations and test papers and their accompanying mark schemes. Mark schemes that do not just award marks for accuracy and for the application of mathematical techniques, but instead reward problem solving processes, will serve to encourage the embedding of problem solving. More open-ended questions will require mark schemes that are less prescriptive than those suitable for questions involving more routine tasks. Inevitably such changes will impact on the setting and marking of examinations and tests. Those involved in these activities require knowledge and understanding of how to assess problem solving effectively (see the recommended actions in section 4).

19 For example, a problem solving question could be phrased as text which includes some information and asks for a recommended course of action. To solve the problem the text might have to be translated into a mathematical equation which has to be solved. Once the solution or solutions to the equation have been obtained these will have to be used to answer the original question – namely to provide a recommended course of action.

4 The next steps for the assessment of problem solving

Problem solving must remain a key policy priority for mathematics assessment.

As noted in section 2, curriculum expectations in mathematics now emphasise the importance of problem solving (see Table 1). To make real progress, these laudable expectations must also be evident in sample assessment and exemplification materials and in assessments themselves.

There is also substantial room for further improvements in assessment to ensure that the problem solving skills in mathematics that are so highly valued by employers and universities are developed by all young people.

4.1 Recommended actions

Some key actions are urgently required if the reforms are to succeed and if problem solving in assessment is to be improved over time.

(A) Regulation

A1) There must be an emphasis on continuity across phases of education in the implementation of current reforms and any subsequent revisions to mathematics curricula and qualifications. This should be reflected in current and future policy developments.

A2) The Standards and Testing Agency and Ofqual should put in a place a structured plan to ensure comparability and reliability of assessments involving more problem solving questions across all phases of mathematics education.

A3) The Standards and Testing Agency and awarding organisations should be required, and supported, to develop test and examination papers that encourage young people to engage in rich problem solving activities and to develop mark schemes that reward problem solving approaches.

A4) Given the challenges of marking questions with problem solving characteristics, there must be an investment in providing guidance for examiners across all phases and in their professional development.

(B) Research and Development; Monitoring and Evaluation

B1) The current reforms and their implementation should be monitored and evaluated to ensure that the curriculum aims on problem solving are being reflected in assessments at the end of each key stage.

B2) A range of assessment methods need to be developed and trialled to determine how to assess mathematical problem solving most effectively across phases of education in the future. This work should draw on research methodologies and findings.

B3) The assessment of problem solving may require learners to have sufficient time to trial and test out various approaches (see Table 2). Research and development is needed to inform decisions about appropriate time allocations for mathematics examinations and tests which include problem solving.

B4) Mathematics education expertise should be utilised in a strategic and transparent way in the design and review of assessments.

(C) Change and Professional Development

C1) Changes in the assessment of problem solving should be introduced incrementally. A commitment is required across the community to the gradual increase in the quality of problem solving assessment and a reduction in scaffolding over time. An incremental approach will allow time for schools, teachers and learners to adapt to the changes. It will also enable teachers to undertake the necessary professional development required to support sustainable changes in the teaching and learning of problem solving in the classroom.

5 Conclusion

If all young people are to successfully progress into university or employment well prepared for the mathematical and quantitative demands of the modern world, there is a clear need for improvements in the teaching, learning and assessment of problem solving across all phases.

ACME welcomes the strong emphasis on problem solving in recent reforms. With careful implementation, monitoring and evaluation of these reforms, progress can be made in the assessment problem solving.

This will require a sustained commitment to incremental improvement across all phases of mathematical education. It will be most successful if there is strong collaboration between the mathematics community, the Department for Education, the Standards and Testing Agency, Ofqual, employers and universities.

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Professor Andrew Noyes, ACME Outer Circle; University of Nottingham

Professor Anne Watson, University of Oxford

Anne White, ACME Deputy Chair; Improve Maths

Royal Society staff (Science Policy Centre, Industry and Design teams)

ACME

For further information about the Advisory Committee on Mathematics Education:

T +44 (0)20 7451 2588
W acme-uk.org
E acme@royalsociety.org
@ACMEmaths

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